Int. J. Solids Structures Vol. 26, No. 5/6, pp. 675–681, 1990
Printed in Great Britain.

0020-7683/90 \$3.00 + .00

INFLUENCE OF LARGE AMPLITUDES ON FREE FLEXURAL VIBRATIONS OF POLYGONAL SHEAR-DEFORMABLE PLATES—A UNIFYING DIMENSIONLESS FORMULATION

HANS IRSCHIK
Institut für Allgemeine Mechanik, Technical University Vienna, Wiedner Hauptstraße 8-10, A-1040 Vienna, Austria

(*Received* 21 *April* 1989; *in revised form* 22 *August* 1989)

Abstract—Using a proper scaling of the amplitude-to-thickness ratio, the influence of large amplitudes on free flexural vibrations of shear-deformable plates with hinged edges and polygonal planforms becomes independent of the special plate geometry. From this non-dimensional formulation, the nonlinear fundamental period is calculated using the Dirichlet–Helmholtz-eigenvalue of the domain under consideration. On the other hand, results from literature for various polygonal plate geometries can be unified by a transformation, taking into account the influence of shear or neglecting this effect.

1. INTRODUCTION

In the classical paper by Chu and Herrmann (1956), a remarkable influence of "large" amplitudes on the periods of flexural vibrations of elastic plates is reported.† Figure 1 shows the results of Chu and Herrmann (1956) for simply supported rectangular plates with immovable edges in the form of frequency ratios $\omega_L/\omega_N = T_N/T_L$ as a function of the "amplitude"-to-thickness ratio c/h. (L) and (N) stand for linearized and nonlinear results, respectively, and T is the fundamental vibration period. It is seen from Fig. 1 that there is little influence of the plate aspect ratio a/b, but the plate strip $a/b \to 0$ does not converge to the simply supported beam with span a.

Those results have been derived following von Karman's nonlinear theory of linear elastic plates rigid in shear, cf. Ziegler (1985, p. 198). Solutions of von Karman's equations, however, are difficult to obtain for even the simplest plate geometries or conditions of support. Considerable improvement has been achieved by Berger (1955), who neglected the influence of the second invariant of the middle surface strains on the plate deformation. Thus, the influence of von Karman's in-plane forces is replaced by tensile forces of hydrostatic type being constant within the whole plate domain. It has been emphasized by Nowinski and Ohnabe (1972) that the corresponding simplified calculations result in an excellent agreement with von Karman's theory as long as in-plane motions of the plate edges are prevented. While this statement is reasonable from a mechanical point of view, a mathematical justification has been given by Schmidt (1974) using a proper perturbation method. Nonlinear free flexural vibrations of rectangular plates with hinged immovable edges have been treated by Nash and Moodier (1960) following Berger's approximate theory. Identical results have been derived by Wah (1963), who achieved further simplifications in the solution strategy by *a priori* eliminating the in-plane displacements from the equations. Results of Nash and Modeer (1960) and Wah (1963), which are also shown in Fig. 1, agree well with those of Chu and Herrmann (1956). Note, however, that the former results are independent of the rectangle's aspect ratio a/b.

This latter observation is the starting point of the present paper: Following Wah's solution strategy, but using an adequate dimensionless formulation of Berger's equations extended to shear-deformable plates, it is shown that the ratio T_N/T_L of polygonal plates with hinged, immovable edges can be presented in a form independent of the special plate geometry. Conversely, the Berger-type results for rectangles given in Fig. 1 can be used—

† A comprehensive recent review on nonlinear plate vibrations is given by Sathyamoorthy (1987).

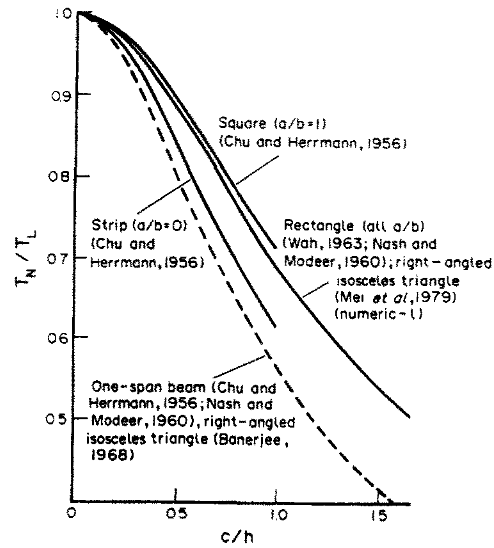

Fig. 1. Dimensionless nonlinear fundamental vibration period T_N/T_L as a function of amplitude-to-thickness ratio c/h for various plate geometries, $s = 0$. (Note that, fixing T_N/T_L, the ratio of c/h for rectangle and one-span beam is $\sqrt{2}$.)

when properly interpreted—for such plates of arbitrary polygonal planform. (The cited discrepancy between one-span beam and plate strip is resolved as a by-product.)

These statements are confirmed by means of results from the literature concerning rectangular and triangular plates.

2. BERGER'S EQUATIONS FOR THE FREE VIBRATIONS OF SHEAR-DEFORMABLE PLATES

Due to Berger's assumption of neglecting the influence of the second invariant of the strain tensor in the middle surface, the dynamic plate deflection w is governed by the differential equation of the linearized problem supplemented by tensile hydrostatic in-plane forces n, which are uniformly distributed within the plate domain

$$K(1+ns)\Delta\Delta w - n\Delta w - K\rho hs\Delta\ddot{w} + \rho h\ddot{w} = 0. \tag{1}$$

In eqn (1), $K = Eh^3/12(1-\nu^2)$ is the plate stiffness, ρ denotes mass density, and Δ is Laplace's operator. A dot stands for time derivative. Influence of shear is given by

$$s = 1/\kappa^2 Gh \tag{2}$$

in case of isotropic plates with the coefficient of shear $\kappa^2 = 5/6$. Plates considered rigid in shear are described by the limit $s \to 0$. The influence of rotatory inertia is neglected in eqn (1), which formally is a specialization of Mindlin's theory of linearized vibrations taking into account in-plane forces n, cf. Irschik (1985). Equation (1) is in accordance with the Berger-type equations of shear-deformable plates given by Wu and Vinson (1969), compare eqn (32) of Wu and Vinson (1969). It has been noted by Wu and Vinson (1969) that the influence of rotatory inertia generally is small for the problem under consideration, while the influence of shear may become important, especially in case of plates made of transversely isotropic material, where G of eqn (2) has to be replaced by the transverse shear modulus $G_c = \sigma_{xz}/2\varepsilon_{xz} = \sigma_{yz}/2\varepsilon_{yz}$. (There is, e.g., $20 \leqslant E/G_c \leqslant 50$ in the case of pyrolytic graphite-type materials, while $E/G = 2(1+\nu)$ in the isotropic case.)

Furthermore, eqn (1) has to be used for sandwich plates with thin surface layers, where the proper expressions for K and s may be taken from the literature, cf. Plantema (1966).

It has been shown by Irschik (1985) within the context of Mindlin's sixth-order theory of free vibrations that the fourth-order differential eqn (1) is sufficient for the determination of the deflection w in the case of simply supported plates of polygonal planform. In this latter case, the boundary conditions of vanishing deflection, edge rotation and bending moment can be decoupled to two conditions in terms of w, namely to Navier's form

$$w = 0, \quad \Delta w = 0 \text{ at the boundary} \tag{3}$$

well-known from plates rigid in shear (Ziegler, 1985, p. 281).

Following Wu and Vinson (1969), Berger's normal force n is not affected by the influence of shear explicitly. Hence, according to Wah (1963)

$$n = -\frac{D}{2A}\int_A w\Delta w \, \mathrm{d}A \tag{4}$$

where $D = Eh/(1-\nu^2)$, and A is the area of the plate domain. Equation (4) is valid, if the plate edges are prevented from in-plane motions. This case of immovable edges is the appropriate one for Berger's theory, cf. Nowinski and Ohnabe (1972), Schmidt (1974).

Equation (4), the differential eqn (1) and the boundary conditions (3) altogether with proper initial conditions define the nonlinear initial-boundary-value problem under consideration.

3. SOLUTION IN NON-DIMENSIONAL FORM

Following Chu and Herrmann (1956), solutions of the problem are sought in the separated form (single-term Ritz–Ansatz):

$$w(\vec{x};t) = q(t)w^*(\vec{x}) \tag{5}$$

where the non-dimensional generalized coordinate q is due to appropriate initial conditions, e.g.

$$q(0) = 1, \quad \dot{q}(0) = 0. \tag{6}$$

In order to study the influence of large amplitudes upon the fundamental period, w^* is set proportional to the first eigenfunction of the linearized theory w_1, and is scaled with the amplitude c, which carries the dimension of length

$$w^* = cw_1. \tag{7}$$

Note that c is not necessarily equal to max (w^*). Applying Galerkin's procedure to eqn (1) and inserting eqn (5) into eqn (4) leads to the following equation of motion for the generalized coordinate q

$$\ddot{q} + \frac{K(1+ns)\int_A w_1\Delta\Delta w_1 \, \mathrm{d}A - n\int_A w_1\Delta w_1 \, \mathrm{d}A}{\rho h\int_A w_1^2 \, \mathrm{d}A - K\rho hs\int_A w_1\Delta w_1 \, \mathrm{d}A} q = 0 \tag{8}$$

where

$$n = -\frac{D}{2A}c^2q^2\int_A w_1\Delta w_1 \, \mathrm{d}A. \tag{9}$$

In order to get general results, the integrals occurring in eqns (8) and (9), which contain second- as well as fourth-order derivatives of w_1, have to be determined independent of the special planform of the plate. Such general expressions exist for the case of polygonal, simply supported plates, where—due to boundary conditions (3)—the eigenfunction w_1 is governed by the Dirichlet–Helmholtz-eigenvalue problem

$$\Delta w_1 + \alpha_1 w_1 = 0$$
$$w_1 = 0 \text{ at the boundary} \tag{10}$$

with α_1 denoting the first eigenvalue. Note that eqn (10) is valid for such plates with or without in-plane forces, and that it is valid for Kirchhoff-plates (rigid in shear), cf. Ziegler (1985), p. 437, as well as in Mindlin's theory of shear-deformable plates. The latter point has been worked out in Irschik (1985), where the corresponding relations between α_1 and $T_L = 2\pi/\omega_{1L}$ are listed.

Using eqn (10), which reflects the eigenvalue problem of a linear elastic prestressed membrane, and indeed is restricted to the plate type under consideration, all the integrals in eqns (8), (9) can be expressed by means of the membrane eigenvalue α_1

$$\int_A w_1 \Delta w_1 \, \mathrm{d}A = -\alpha_1 \tilde{\beta}_1 A \tag{11}$$

$$\int_A w_1 \Delta\Delta w_1 \, \mathrm{d}A = \alpha_1^2 \tilde{\beta}_1 A \tag{12}$$

where the norm $\tilde{\beta}_1$ is defined through

$$\int_A w_1^2 \, \mathrm{d}A = \tilde{\beta}_1 A. \tag{13}$$

Thus, eqn (8) becomes

$$\ddot{q}[\rho h(1+Ks\alpha_1)/K\alpha_1^2] + q + 6\tilde{\beta}_1 \frac{c^2}{h^2}(1+Ks\alpha_1)q^3 = 0. \tag{14}$$

Equation (14) is equivalent to the dimensionless formulation

$$q'' + q + \tilde{c}q^3 = 0 \tag{15}$$

where the non-dimensional amplitude parameter $\tilde{c}$ is

$$\tilde{c} = 6\tilde{\beta}_1 \frac{c^2}{h^2}(1+Ks\alpha_1) \tag{16}$$

dimensionless time is

$$\tilde{t} = [K\alpha_1^2/\rho h(1+Ks\alpha_1)]^{1/2} t \tag{17}$$

and a prime denotes differentiation with respect to $\tilde{t}$.

The exact solution of this homogeneous Duffing eqn (15) under initial conditions (6) is given in terms of Jacobi's elliptic cosine-function

$$q = cn(u|m) \tag{18}$$

where

$$u = (1+\tilde{c})^{1/2}\tilde{t} \tag{19}$$

$$m = \tilde{c}/2(1+\tilde{c}). \tag{20}$$

For initial conditions $q(0) = 0$, $\dot{q}(0) = 1$, the solution is proportional to the elliptic function *sd*. The elliptic functions *cn* and *sd* pass over to the transcendental cosine and sine functions, respectively, if $\tilde{c} \to 0$. Note that the corresponding linearized period of vibration is

$$\tilde{T}_L = 2\pi \tag{21}$$

whereas both elliptic functions have the period

$$\tilde{T}_N = 4k(m)/(1+\tilde{c})^{1/2}. \tag{22}$$

k is the complete elliptic integral of the first kind.

The ratio of nonlinear to linearized fundamental period is equal in non-dimensional and in real-time representation

$$\frac{T_N}{T_L} = \frac{\tilde{T}_N}{\tilde{T}_L} = \frac{2k(m)}{\pi(1+\tilde{c})^{1/2}}. \tag{23}$$

Analogously, using the Ritz–Ansatz for the periodic time function

$$q \simeq \cos(2\pi\tilde{t}/\tilde{T}_N) \tag{24}$$

and applying Galerkin's procedure to eqn (15), integration is over one period, the approximation

$$\frac{T_N}{T_L} = \frac{\tilde{T}_N}{\tilde{T}_L} \simeq (1+3\tilde{c}/4)^{-1/2} \tag{25}$$

to eqn (23) is derived, which, as has been demonstrated by Chu and Herrmann (1956) in the case of a plate strip, is of sufficient accuracy within the range of $\max(w^*)/h < 5/2$.

4. FORMULATION OF THE ANALOGY

Equations (23) and (25) reflect the main result of the present paper:

(i) Plates with polygonal, simply supported, immovable edges having the same value of $\tilde{c}$ result in the same ratio T_N/T_L, regardless of the special geometry of their planform. (This has been shown above within the context of Berger's theory including the effect of shear). T_N/T_L is given in Fig. 2 as a function of $\tilde{c}$.

In literature, however, T_N/T_L is not given as a function of the similarity complex $\tilde{c}$ of eqn (16), but as a function of c/h alone. Since the norm $\tilde{\beta}_1$ may be chosen freely, such a known result from literature for a special plate geometry can be transformed to another problem:

(ii) Given T_N/T_L for the triple of values c/h, $\tilde{\beta}_1$ and $Ks\alpha_1$, a plate with the value $(Ks\alpha_1)^*$ results in this same ratio T_N/T_L at the same ratio c/h, if the norm $\tilde{\beta}_1^*$ is chosen to be

$$\tilde{\beta}_1^* = \tilde{\beta}_1(1+Ks\alpha_1)/(1+Ks\alpha_1)^* \tag{26}$$

i.e. if w^* of eqn (7) is rescaled in the proper manner.

(iii) Conversely, given T_N/T_L for a given $\tilde{c}$, a plate with $(Ks\alpha_1)^*$ and $\tilde{\beta}_1^*$ results in the

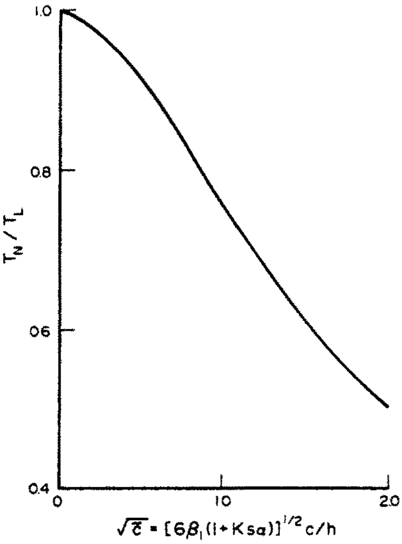

Fig. 2. Unifying representation of T_N/T_L as a function of the dimensionless amplitude parameter $(\tilde{c})^{1/2}$, eqn (16), including the effect of shear. T_L denotes the linear period of the shear-deformable plate. (For example: quadratic plate, $E/G_c = 50$, $\nu = 0.25$, $\alpha_1 = 2\pi^2/a^2$, $\tilde{\beta}_1 = 1/4$, $c/h = 1$, $\sqrt{\tilde{c}} \doteq 1.75$, $T_N/T_L \doteq 0.55$; from Wu and Vinson (1969) Figure 1, cf. eqn (45): $T_N/T_L \doteq 1/(1.26\sqrt{2.05}) = 0.55$).

same value of T_N/T_L at the ratio

$$(c/h)^* = [\tilde{c}/6\tilde{\beta}_1^*(1+(Ks\alpha_1)^*)]^{1/2}. \tag{27}$$

As a consequence, results from plates rigid in shear ($s = 0$) can be transformed to shear-deformable plates.

Those transformations, eqns (26) and (27), respectively, may serve as control conditions, while the main unifying representation is shown in Fig. 2, using the proper similarity complex $\tilde{c}$ of eqn (16), taking into account a free choice of the norm $\tilde{\beta}_1$† and the physical meaning of the "amplitude"-scale $\tilde{c}$, respectively.

In a first example, the discrepancy between the results of Wah (1963) for rectangular plates and the results for one-span beams, see Fig. 1, is resolved. In Wah (1963), $w_1 = \sin(\pi x/a)\sin(\pi y/b)$ has been used. Thus, $\tilde{\beta}_1 = 1/4$ and $\tilde{c} = 3(c/h)^2/2$. Inserting into eqn (23) with $s = 0$, coincidence with eqn (22) of Wah (1963) (case "a" of Appendix A), is obtained. Results for the one-span beam are derived using $w_1 = \sin(\pi x/a)$. Thus, $\tilde{\beta}_1^* = 1/2$ and, following eqn (27), equal values of T_N/T_L for the rectangle and for the beam should be obtained at $(c/h)^* = (c/h)/\sqrt{2}$, respectively. This is easily checked from Fig. 1.

Next, shear-deformable rectangular plates are considered. The linearized eigenvalue is $\alpha_1 = \pi^2[(1/a^2)+(1/b^2)]$. Using $\tilde{\beta}_1 = 1/4$, $s = 6/5G_c h$, and inserting into eqn (23), coincidence with Figs 1–3 of Wu and Vinson (1969) is obtained.

As a further example, consider right-angled isosceles triangular plates with $s = 0$. Using $w_1 = \sin(\pi x/a)\sin(2\pi y/a)+\sin(2\pi x/a)\sin(\pi y/a)$, there is $\tilde{\beta} = 1/2$. Thus $T_N/T_L = 2k/\pi(1+3(c/h)^2)^{1/2}$, see eqn (23). The same result has been derived by Banerjee (1968). This result is equivalent to the beam-result of Fig. 1, where $\tilde{\beta}_1 = 1/2$. Very close agreement is found with a von Karman-type calculation of Chaudhuri (1982). At a first glance no agreement is found with an advanced FEM-calculation of Mei *et al.* (1979). However, rescaling w_1 such that $w_1(a/4, a/4) = 1$, there is $\tilde{\beta}_1^* = 1/4$, and practically exact agreement of Mei *et al.* (1979) with eqn (23) is the result. Thus Mei's results for triangles

† Conveniently, but not necessarily $\tilde{\beta}_1$ is chosen to give $\max(w^*) = c$. As will be seen below, this is not uniformly performed in literature.

coincides with the results of Wah (1963) for rectangular plates, see Fig. 1. Note from this example that the norm $\tilde{\beta}_1$ has to be introduced explicitly in order to get reproducible results.

5. CONCLUDING REMARKS

It is emphasized that, according to the above analogy, results of Fig. 1 are meaningful not only for rectangular or triangular domains. Furthermore, using the approximation (25), the ratio T_N/T_L may be calculated with the help of a standard pocket computer within a few minutes using the similarity complex $\tilde{c}$ of eqn (16). In case of $s \neq 0$, the Helmholtz eigenvalue α_1 has to be known for an evaluation as a function of c/h, i.e. only a linear second-order boundary value problem has to be solved, likewise to the associated linearized fundamental period. In case of $s = 0$, a first short account of the above results has been given by Irschik (1987).

Acknowledgements—Support of the Austrian "Fonds zur Förderung der wissenschaftlichen Forschung", central project S30/03 is gratefully acknowledged. This work was performed at the Technical University of Vienna, Institut für Allgemeine Mechanik (Head: Prof. Dr. Franz Ziegler).

REFERENCES

Banerjee, B. (1968). Large amplitude free vibration of an isosceles right-angled triangular plate with simply supported edges. *J. Sci. Technol. India* **12,** 52–54.

Berger, H. M. (1955). A new approach to the analysis of large deflection of plates. *J. Appl. Mech.* **22,** 465–472.

Chaudhuri, S. K. (1982). Large amplitude free vibrations of a right-angled isosceles triangular plate of simply supported edges. *J. Sound Vib.* **84,** 81–85.

Chu, H. N. and Herrmann, G. (1956). Influence of large amplitudes of free flexural vibrations of rectangular elastic plates. *J. Appl. Mech.* **23,** 532–540.

Irschik, H. (1985). Membrane-type eigenmotions of Mindlin plates. *Acta Mechanica* **55,** 1–20.

Irschik, H. (1987). Zum Einfluß großer Amplituden auf freie Plattenschwingungen. *ZAMM* **67,** T85–T86.

Mei, C., Narayanaswami, R. and Venkateswara Rao, G. (1979). Large amplitude free flexural vibrations of thin plates of arbitrary shape. *Computers and Structures* **10,** 675–681.

Nash, W. A. and Modeer, J. R. (1960). Certain approximate analyses of the nonlinear behavior of plates and shallow shells. *Proc. Symposium on the Theory of Thin Elastic Shells.* Interscience, New York.

Nowinski, J. L. and Ohnabe, H. (1972). On certain inconsistencies in Berger equations for large deflections of elastic plates. *Int. J. Mech. Sci.* **14,** 165–170.

Plantema, F. J. (1966). *Sandwich Construction.* John Wiley, New York.

Sathyamoorthy, M. (1987). Nonlinear vibration analysis of plates: a review and survey of current developments. *Appl. Mech. Rev.* **40,** 1553–1561.

Schmidt, R. (1974). On Berger's method in the nonlinear theory of plates. *J. Appl. Mech.* **41,** 521–523.

Wah. T. (1963). Large amplitude flexural vibrations of rectangular plates. *Int. J. Mech. Sci.* **5,** 425–438.

Wu, C.-I. and Vinson, J. R. (1969). Influences of large amplitudes, transverse shear deformation and rotatory inertia on lateral vibrations of transversely isotropic plates. *J. Appl. Mech.* **36,** 254–260.

Ziegler, F. (1985). *Technische Mechanik der festen und flüssigen Körper.* Springer, Wien–New York.